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On the stability and spectrum of non-supersymmetric AdS_5 solutions of M-theory compactified on Kähler-Einstein spaces

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Abstract: Eleven-dimensional supergravity admits non-supersymmetric solutions of the form $AdS_5 \times M_6$ where M_6 is a positive Kähler-Einstein space. We show that the necessary and sufficient condition for such solutions to be stable against linearized bosonic supergravity perturbations can be expressed as a condition on the spectrum of the Laplacian acting on $(1, 1)$-forms on M_6. For $M_6 = CP^3$, this condition is satisfied, although there are scalars saturating the Breitenlöhner-Freedman bound. If M_6 is a product $S^2 \times M_4$ (where M_4 is Kähler-Einstein) then there is an instability if M_4 has a continuous isometry. We show that a potential non-perturbative instability due to 5-brane nucleation does not occur. The bosonic Kaluza-Klein spectrum is determined in terms of eigenvalues of operators on M_6.

Keywords: Flux compactifications, AdS-CFT Correspondence, M-Theory

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Contents

1 Introduction

Eleven-dimensional supergravity admits well-known "Freund-Rubin" compactifications of the form $AdS_4 \times M_7$ or $AdS_7 \times M_4$, where M_7 and M_4 are positive Einstein manifolds [1].

Less well-known is the fact that there are also solutions of the form $AdS_5 \times M_6$ where M_6 is a six dimensional positive Kähler-Einstein space [2]. The solutions have metric[1]

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n, \tag{1.1}$$

where $g_{\mu\nu}$ and g_{mn} are the metrics on AdS_5 and M_6 respectively, with Ricci tensors

$$R_{\mu\nu} = -2c^2 g_{\mu\nu}, \qquad R_{mn} = 2c^2 g_{mn}, \tag{1.2}$$

so the radius of AdS_5 is $\ell = \sqrt{2}/|c|$. The 4-form is

$$F = cJ \wedge J, \tag{1.3}$$

where J is the Kähler form on M_6. Examples of suitable M_6 are: CP^3; the quotient SU(3)/T where T is the maximal torus of SU(3); the Grassmanian $Gr_2(R^5)$; or a product[2] $M_4 \times S^2$ where the only possible M_4 are CP^2, $S^2 \times S^2$, or a del Pezzo surface dP_k, $k = 3 \dots 8$ [4, 5]. This list includes all cases for which M_6 is either homogeneous or a product (inhomogeneous non-product M_6 also exist [4]). These solutions are not supersymmetric: for $M_6 = CP^3$ this was proved in [6], and for general M_6 it follows from the analysis of supersymmetric AdS_5 solutions of [7].

By the AdS/CFT correspondence [8], these solutions should be dual to conformal field theories in four dimensions. Flux quantization renders c discrete. For $M_6 = CP^3$, the central charge of the CFT dual to these solutions scales as N^3, where N is the number of units of flux on $CP^2 \subset CP^3$ [8]. This suggests that these solutions may have an interpretation in terms of M5-branes wrapping a 2 cycle. The supergravity approximation is valid for large N.

The purpose of this paper is to investigate the stability of these solutions. We shall examine three potential instabilites. First, we check whether there is a non-perturbative instability due to quantum nucleation of M5-branes (wrapping a 2-cycle in M_6) [9, 10]. We find that this does not happen for any M_6: the 5-brane (Euclidean) action is always positive and an instanton describing such a process never exists.

Secondly, we consider perturbative stability by considering linearized supergravity perturbations. We determine the full bosonic Kaluza-Klein (KK) spectrum for general M_6 in terms of eigenvalues of differential operators on M_6. The gauge group is $G \times \mathrm{U}(1)^{b_2-1}$, where G is the isometry group of M_6 and b_2 the second Betti number of M_6. The squared masses of all fields are non-negative except possibly for scalars arising from $(1,1)$-forms on M_6. Demanding that such modes respect the Breitenlöhner-Freedman (BF) stability bound [11] gives a criterion for stability of these solutions valid for general M_6. Analogous results for Freund-Rubin compactifications of the form $AdS_4 \times M_7$ were obtained in [12], and for Freund-Rubin compactifications of other theories in [13].

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[1] Our conventions are summarized in appendix A.

[2] In the case in which M_6 is a product of lower-dimensional Kähler-Einstein spaces, i.e., $M_6 = M_4 \times S^2$, these solutions can be generalized by taking $F = c_4 J^{(4)} \wedge J^{(4)} + c_2 J^{(4)} \wedge J^{(2)}$, where $J^{(4)}$, $J^{(2)}$ are the Kähler forms on M_4 and S^2 respectively. This gives a 2-parameter family of solutions with independent radii for M_4 and S_2 [3]. Similarly, if $M_6 = S^2 \times S^2 \times S^2$ then one can obtain a 3-parameter family. We shall not consider these generalizations further.

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M_6	Isometry group	Classically stable?
CP^3	SU(4)	yes
$S^2 \times S^2 \times S^2$	$\mathrm{SO}(3)^3$	no
$S^2 \times CP^2$	$\mathrm{SO}(3) \times \mathrm{SU}(3)$	no
$S^2 \times dP_3$	$\mathrm{SO}(3) \times \mathrm{U}(1)^2$	no
$S^2 \times dP_{k>3}$	SO(3)	?
$\mathrm{SU}(3)/T$	SU(3)	?
$Gr_2(R^5)$	SO(5)	?
...	...	...

Table 1. Classical linearized stability results for particular M_6

Our criterion is as follows. Consider transverse, primitive,[3] $(1,1)$-form eigenfunctions of the Hodge-de Rham Laplacian on M_6 with eigenvalue $\lambda_{(1,1)}$. A Kähler-Einstein compactification $AdS_5 \times M_6$ suffers a linearized bosonic *instability* if, and only if, there is a mode with

$$2c^2 < \lambda_{(1,1)} < 6c^2. \tag{1.4}$$

We have investigated the spectrum for some of the M_6 listed above. The results are given in table 1. For $M_6 = CP^3$, the lowest eigenvalue is $\lambda_{(1,1)} = 6c^2$. Hence $AdS_5 \times CP^3$ is stable at the linearized level in classical supergravity. However, there are scalar fields that saturate the BF bound. Therefore an analysis of finite N corrections to the mass would be required to make a definite statement about perturbative stability.[4] The scalars saturating the bound transform in the $[0,2,0]$ representation of SU(4).

For $M_6 = S^2 \times M_4$, one might expect an instability corresponding to the S^2 increasing in radius and M_4 decreasing (or vice versa) since this is what happens for product space Freund-Rubin compactifications [12]. However, such a mode corresponds to $\lambda_{(1,1)} = 0$, and is therefore stable: the flux on the internal space stabilizes the solution against this kind of deformation. However, we find that there is a mode with $\lambda_{(1,1)} = 4c^2$ whenever M_4 possesses a continuous isometry. This implies that $S^2 \times S^2 \times S^2$, $S^2 \times CP^2$ and $S^2 \times dP_3$ give unstable solutions. However dP_k has no continuous isometries for $k > 3$ [15], so the classical stability of $S^2 \times dP_k$ for $k > 3$ requires further investigation.

It would be interesting to determine the spectrum for the other homogeneous spaces $Gr_2(R^5)$ and $\mathrm{SU}(3)/T$. We note that $\mathrm{SU}(3)/T$ possesses a primitive harmonic $(1,1)$-form, so the lowest eigenvalue is $\lambda_{(1,1)} = 0$ in this case, as for the product spaces.

The third possible instability that we have considered is the possibility that quantum corrections could generate a tadpole for a massless, uncharged, scalar field, resulting in runaway behaviour [14]. To examine this possibility, we need to investigate whether there are massless scalars transforming as singlets under G (as no fields are charged under $\mathrm{U}(1)^{b_2-1}$).

[3] "Primitive" means that the contraction with the Kähler form vanishes.

[4] These corrections are of two types. Higher derivative corrections give contributions scaling as powers of $1/N$. Quantum loop corrections give contributions scaling as powers of $1/N^3$.

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A massless scalar will be present if M_6 admits complex structure moduli. Now, dP_k has such moduli for $k > 4$ [15]. Hence $M_6 = S^2 \times dP_k$ has such moduli. These are trivially singlets under G (since dP_k has no continuous symmetries for $k \geq 4$). Therefore we conclude that no symmetry prevents quantum corrections from destabilizing compactifications with $M_6 = S^2 \times dP_k$ for $k > 4$, at least at a generic point in moduli space (at special points there may be discrete symmetries preventing this from happening). Clearly this can happen whenever M_6 has complex structure moduli invariant under G, in particular if M_6 has complex structure moduli and no isometries.

This paper is organized as follows. In section 2, we give a detailed summary of our results. We first investigate quantum nucleation of M5-branes. We then summarize our analysis of the Kaluza-Klein spectrum, explain the origin of our stability criterion, and investigate this criterion for several possible M_6. Section 3 contains the full calculation of the Kaluza-Klein spectrum.

2 Results

2.1 5-brane nucleation

A potential non-perturbative instability involves quantum nucleation of branes [9, 10]. Since the solutions are purely magnetic, we need only consider nucleation of 5-branes. The (Euclidean) 5-brane action is

$$S = T \int d^6\xi \sqrt{h} - T \int C_{(6)}, \tag{2.1}$$

where T is the 5-brane tension, ξ are worldvolume coordinates, h the determinant of the induced metric on the worldvolume and $C_{(6)}$ the 6-form potential for $\star F$.[5] For the solutions of interest, $\star F = 2c\eta_5 \wedge J$, where η_5 is the volume form of AdS_5. We are looking for instanton solutions so we work in Euclidean signature, writing the metric on Euclidean AdS_5 as

$$ds^2 = d\rho^2 + \ell^2 \sinh^2(\rho/\ell) d\Omega_4^2. \tag{2.2}$$

We can choose the gauge ($\ell = \sqrt{2}/|c|$)

$$C_{(6)} = \frac{8}{c^3} \left[\int_0^\rho \sinh^4(cx/\sqrt{2}) dx \right] d\Omega_4 \wedge J. \tag{2.3}$$

To get a non-trivial contribution from the flux term in the action, we take the 5-brane worldvolume to be $S^4 \times \Sigma$ where S^4 is a sphere of constant ρ in AdS_5 and Σ a 2-cycle in M_6. Upon continuing to Lorentzian signature this would give an exponentially expanding 5-brane with worldvolume $dS_4 \times \Sigma$. Evaluating the action on this Ansatz gives

$$S = \frac{4T}{c^4} \Omega_4 \left[\sinh^4(c\rho/\sqrt{2}) V - 2c \int_0^\rho \sinh^4(cx/\sqrt{2}) dx \int_\Sigma J \right], \tag{2.4}$$

[5] Note that the M-theory Chern-Simons term vanishes for these solutions so there are no subtleties in defining $C_{(6)}$.

λ	$\lambda_{(1)}$	$\lambda_{(1,1)}$	$\lambda_{(2,0)}$	$\lambda_{(2,1)}$
$c^2k(k+3)$	$c^2(k+2)(k+4)$	$c^2(k+2)(k+3)$	$c^2(k+3)(k+4)$	$c^2(k+2)(k+4)$
$[k,0,k]$	$[k,1,k+2]$	$[k,2,k]$	$[k,0,k+4]$	$[k,1,k+2]$

Table 2. Eigenvalues of the Laplacian on CP^3 acting on transverse primitive forms, determined from [19]. k is a non-negative integer. $\lambda \equiv \lambda_{(0,0)}$, $\lambda_{(1)} \equiv \lambda_{(1,0)}$. There are no transverse $(3,0)$-forms. The bottom row gives the corresponding representation of SU(4). If a (p,q)-form eigenfunction belongs to representation $[r,s,t]$ then the (q,p)-form eigenfunction belongs to the complex conjugate representation $[t,s,r]$.

where V is the volume of Σ. Varying with respect to ρ gives the condition for a turning point (for $c>0$)

$$\tanh(c\rho/\sqrt{2}) = \frac{\sqrt{2}V}{\int_\Sigma J} \geq \sqrt{2}, \tag{2.5}$$

where the inequality follows from the fact that J is a calibration in M_6. Hence there is no solution for ρ (the action is positive and monotonically increasing with ρ) so we conclude that there is no 5-brane nucleation instability.

It would be interesting to investigate more complicated non-perturbative instabilities, such as the one of [16], which involves simultaneous nucleation of branes and a Kaluza-Klein bubble. However, since M_6 must be simply connected [17], our spacetimes do not contain a circle that can collapse to zero size at a bubble. Perhaps there could be an instability involving a bubble describing the collapse of a higher-dimensional submanifold of spacetime, e.g. an S^2 inside M_6.

2.2 The Kaluza-Klein spectrum

2.2.1 Harmonics on M_6

To determine the KK spectrum, we expand each field in terms of harmonics on M_6. These harmonics satisfy various conditions. In particular, we will be concerned with (p,q)-form eigenfunctions of the Hodge-de Rham Laplacian

$$\Delta_6 \hat{Y}_{(p,q)} = \lambda_{(p,q)} \hat{Y}_{(p,q)}, \tag{2.6}$$

which are *primitive*:

$$J^{mn} \hat{Y}_{(p,q)mn\ldots} = 0, \tag{2.7}$$

and *transverse*:

$$d_6^\dagger \hat{Y}_{(p,q)} = 0. \tag{2.8}$$

A hat on a (p,q)-form will be used to denote that it is primitive and transverse. As we explain below, a general (p,q)-form can be decomposed into a primitive, transverse piece and pieces built from forms of lower rank.

For CP^N, the spectrum of the Laplacian acting on (p,q) forms was determined in [19]. Using these results, one can determine the eigenvalues of the Laplacian acting on transverse primitive forms on CP^3. These are summarized in table 2.

We recall a few facts about eigenfunctions of the Hodge-de Rham Laplacian on general M_6 [17, 18]. There are no harmonic $(p,0)$-forms so $\lambda_{(p,0)} = \lambda_{(0,p)} > 0$. In particular, this implies there are no harmonic 1-forms. It also implies that there are no transverse $(3,q)$-forms since such forms would be annihilated by both ∂ and $\partial^\dagger$, and hence be harmonic. For scalars, which we shall take to be real, non-constant eigenfunctions have $\lambda \geq 4c^2$. Eigenfunctions saturating the bound are in one-to-one correspondence with Killing vector fields. This is because a vector field V on M_6 is Killing if, and only if, it can be written as $d_6^c Y$ where Y is a scalar eigenfunction with $\lambda = 4c^2$.

2.2.2 (1,1)-form perturbations

We perform a full linearized analysis of the bosonic Kaluza-Klein spectrum in section 3. The result of this analysis is that the only modes that could violate the Breitenlöhner-Freedman stability bound, indeed the only modes with negative squared mass, arise from $(1,1)$-forms on M_6. These are associated with hermitian metric perturbations on M_6 (i.e. perturbations for which, in complex coordinates, the zz and $\bar{z}\bar{z}$ components of the metric perturbation vanish). Explicitly, the metric perturbation is

$$\delta g_{mn}(x,y) = -\sum_I h^I(x) \hat{Y}^I_{(1,1)mp}(y) J^p{}_n. \tag{2.9}$$

Here we have performed the usual Kaluza-Klein decomposition of modes into a product of fields in AdS_5 and M_6. The former are the scalars $h^I(x)$. On M_6, $\hat{Y}^I_{(1,1)}$ denotes a primitive, transverse, $(1,1)$-form eigenfunction of the Hodge-de Rham Laplacian, with eigenvalue $\lambda^I_{(1,1)}$:

$$\Delta_6 \hat{Y}^I_{(1,1)} = \lambda^I_{(1,1)} \hat{Y}^I_{(1,1)}. \tag{2.10}$$

Modes with different I will decouple from each other. We shall suppress the I index in what follows.

This metric perturbation will couple to terms in the 4-form perturbation that also arise from $(1,1)$-forms on M_6. These are of the form

$$\delta F = d\left(k^-(x) d_6^c \hat{Y}_{(1,1)}(y)\right). \tag{2.11}$$

We can take $\hat{Y}_{(1,1)}$ to be real hence h and k^- are real.

For these modes, the perturbed Maxwell equation reduces to

$$(\Delta + \lambda_{(1,1)})k^- - 4ch = 0 \qquad \lambda_{(1,1)} \neq 0. \tag{2.12}$$

The restriction $\lambda_{(1,1)} \neq 0$ arises from the fact that if $Y_{(1,1)}$ is harmonic then $d_6^c Y_{(1,1)}$ vanishes hence k^- is unphysical. The perturbed Einstein equation reduces to

$$(\Delta + \lambda_{(1,1)} + 4c^2)h - 4c\lambda_{(1,1)}k^- = 0. \tag{2.13}$$

Hence if $\lambda_{(1,1)} = 0$ then we have a single physical real scalar field $h(x)$ with $m^2 = 4c^2$.

However, if $\lambda_{(1,1)} > 0$ then we have two fields and we need to diagonalize the above equations to determine the mass spectrum. Doing so, we find the masses are given by

$$m_\pm^2 = \lambda_{(1,1)} + 2c^2 \pm \sqrt{16c^2\lambda_{(1,1)} + 4c^4}. \tag{2.14}$$

m_+^2 is positive but m_-^2 may be negative. An instability occurs if the Breitenlöhner-Freedman bound is violated, i.e., $m_-^2 < -2c^2$. This is equivalent to

$$2c^2 < \lambda_{(1,1)} < 6c^2 \qquad \text{for instability.} \tag{2.15}$$

If there exists a (primitive, transverse) $(1,1)$-form eigenfunction of the Laplacian on M_6 with eigenvalue in this range then the solution is unstable.

2.2.3 Stability of CP^3

The results of table 2 give

$$m_+^2 = c^2(k+3)(k+6), \qquad m_-^2 = c^2(k-1)(k+2). \tag{2.16}$$

Hence $m_-^2 \geq -2c^2$ so the Breitenlöhner-Freedman bound is respected. However, modes with $k = 0$ give scalar fields that can saturate the bound. These fields transform in the $[0,2,0]$ representation of SU(4). Since there is no supersymmetry to protect the masses, it is necessary to examine whether higher derivative corrections (corresponding to finite N corrections in the dual CFT) raise or lower the masses of these fields in order to make a conclusive statement about perturbative stability.

Note that there are also massless fields arising from modes with $k = 1$, in the $[1,2,1]$ of SU(4). Since these are charged under the SU(4) isometry group, a runaway associated with these fields is not expected [14].

The dimensions of CFT operators dual to the fields arising from $(1,1)$-forms on CP^3 are generically irrational (the special $k = 0,1$ fields just mentioned excepted).

2.2.4 Instability of $S^2 \times M_4$

In Freund-Rubin compactifications, there is generically an instability if the internal space is a product [1, 12]. The instability arises from a metric deformation of the internal space in which one factor in the product expands and the other contracts. For product space Kähler-Einstein compactifications, we shall see that this simple instability is absent but there is a more complicated instability, at least if M_4 has a continuous isometry.

Assume that $M_6 = S^2 \times M_4$ where M_4 is Kähler-Einstein. The Freund-Rubin product instability arises from (transverse, traceless) metric perturbations of the form

$$\delta g_{mn} \propto h(x)(2g^{(2)}_{mn} - g^{(4)}_{mn}), \tag{2.17}$$

where $g^{(2,4)}_{mn}$ are the metrics of S^2 and M_4 respectively. This is equivalent to a $(1,1)$-form perturbation for which

$$\hat{Y} \propto 2J^{(2)} - J^{(4)}, \tag{2.18}$$

where $J^{(2,4)}$ are the Kähler forms of S^2 and M_4 respectively (so $J = J^{(2)} + J^{(4)}$). The relative factor in the above equation is fixed by the primitivity condition. However, these are covariantly constant hence $\hat{Y}$ is harmonic, i.e., $\lambda_{(1,1)} = 0$, so these modes do *not* lie within the "window of instability" of equation (2.15): they are stable. The presence of flux on the internal space stabilizes it against this kind of deformation.

To obtain an instability we need to look at more complicated modes. Consider $M_6 = S^2 \times S^2 \times S^2$ (i.e. $M_4 = S^2 \times S^2$). Let y_i be coordinates, and $J^{(i)}$ the Kähler form, of the ith S^2. Let Y be a $\lambda = 4c^2$ scalar eigenfunction on S^2, which must exist because S^2 admits Killing vector fields. Now consider the following primitive, transverse, $(1,1)$-form on M_6:

$$\hat{Y} = (Y(y_2) - Y(y_3))\, J^{(1)}(y_1) + (Y(y_3) - Y(y_1))\, J^{(2)}(y_2) + (Y(y_1) - Y(y_2))\, J^{(3)}(y_3). \quad (2.19)$$

A calculation reveals that this is an eigenfuction of Δ_6 with eigenvalue $\lambda_{(1,1)} = 4c^2$, i.e., a mode within the range (2.15). Hence $M_6 = S^2 \times S^2 \times S^2$ is an unstable compactification.

A similar construction works whenever M_4 admits a Killing vector field. Let Y be a scalar harmonic on M_4 with eigenvalue λ. From this we can build a suitable $(1,1)$-form by considering an arbitrary linear combination of $d_4 d_4^c Y$, $Y J^{(4)}$ and $Y J^{(2)}$ (where d_4 is the exterior derivative on M_4), and fixing the coefficients by demanding primitivity and transversality. This gives

$$\hat{Y} = d_4 d_4^c Y - \lambda Y J^{(4)} - 2\lambda Y J^{(2)}. \quad (2.20)$$

This is a $(1,1)$-form eigenfunction of Δ_6 with eigenvalue λ. If M_4 admits a Killing vector field then there exists a mode with $\lambda = 4c^2$ and hence, from (2.15), an instability. It follows that the $S^2 \times CP^2$ and $S^2 \times dP_3$ are unstable compactifications. However, the Kähler-Einstein metric on dP_k does not admit continuous symmetries for $k > 3$ [15] so we cannot conclude that $S^2 \times dP_k$ is unstable for $k > 3$ using this method (unless it could be shown that the lowest non-trivial eigenfunction of the scalar Laplacian on dP_k has $\lambda < 6c^2$).

2.2.5 The full bosonic KK spectrum

In section 3 we determine the full spectrum of bosonic KK excitations. The results are summarized in table 3. Note that there are some curious degeneracies between 2-form, 1-form and scalar fields.

For CP^3, plugging in the known eigenvalues of the Laplacian acting on (p,q)-forms (table 2) gives the mass spectrum of table 4. The eigenvalue $\lambda^{(0,1)}_{(1,0)}$ can be determined from the eigenvalue of the Lichnerowicz operator acting on anti-hermition tensor modes (see section 3.5.7). The general form of these eigenvalues in terms of a non-negative integer k is known [20] but the precise lower bound on k is not (i.e. the smallest allowed value of k may be positive).

2.2.6 The massless spectrum

In addition to the AdS_5 graviton, there are massless vector and scalar fields. There is a massless vector for each Killing vector field on M_6 (associated with $\lambda = 4c^2$ scalar harmonics). These are the usual KK gauge bosons. Massless vectors also arise from primitive harmonic $(1,1)$-forms on M_6. These are familiar from Freund-Rubin compactifications [1] except that here we have the primitivity condition. There are $b_2 - 1$ primitive harmonic $(1,1)$-forms hence the gauge group of the effective 5d theory is $G \times \mathrm{U}(1)^{b_2-1}$.

Massless scalar fields need special consideration because, as discussed in the introduction, the presence of uncharged massless scalars may lead to a runaway instability arising

Field	Type	m^2	Restriction	Section
Spin-2	real	λ		3.5.1
2-form	complex	$\left(\sqrt{\lambda_{(1)}+c^2} \pm c\right)^2$	$\lambda_{(1)} > 0$	3.5.2
	real	$\lambda + 4c^2$	$\lambda > 0$	3.5.3
1-form	complex	$\lambda_{(2,0)}$	$\lambda_{(2,0)} > 0$	3.5.4
	real	$\lambda_{(1,1)}$		3.5.4
	complex	$\left(\sqrt{\lambda_{(1)}+c^2} \pm c\right)^2$	$\lambda_{(1)} > 0$	3.5.5
	real	$\lambda + 4c^2$	$\lambda > 0$	3.5.6
	real	$\lambda + 6c^2 \pm \sqrt{(\lambda+6c^2)^2 - \lambda(\lambda - 4c^2)}$	only + if $\lambda = 0$	3.5.6
Scalar	complex	$\lambda_{(2,1)}$		3.5.8
	complex	$\lambda_{(2,0)}$	$\lambda_{(2,0)} > 0$	3.5.9
	real	$\lambda_{(1,1)} + 2c^2 \pm \sqrt{16c^2\lambda_{(1,1)} + 4c^4}$	only + if $\lambda_{(1,1)} = 0$	2.2.2
	complex	$\lambda^{(0,1)}_{(1,0)}$		3.5.7
	complex	$\left(\sqrt{\lambda_{(1)}+c^2} \pm c\right)^2$	$\lambda_{(1)} > 0$	3.5.10
	real	$\lambda + 4c^2$	$\lambda > 0$	3.5.11
	real	$\lambda + 6c^2 \pm \sqrt{(\lambda+6c^2)^2 - \lambda(\lambda - 4c^2)}$	only + if $\lambda = 0$	3.5.11
	real	0 (axion)		3.5.11

Table 3. The bosonic Kaluza-Klein spectrum. M_6 does not admit harmonic $(p,0)$-forms, so $\lambda_{(p,0)} > 0$. The other restrictions in this table arise because the associated modes are unphysical, i.e., give vanishing metric and 4-form perturbations. $\lambda^{(0,1)}_{(1,0)}$ is the eigenvalue of the Laplacian acting on $(1,0)$-forms taking values in the anti-holomorphic tangent space of M_6 (which vanishes for infinitesimal complex structure deformations).

from a tadpole generated by quantum corrections [14]. Massless scalars arise in several ways. First, dualizing the KK zero mode of the M-theory 3-form in AdS_5 gives a scalar axion. Classically, this field has a continuous shift symmetry. However, quantum mechanically, the axion may develop a potential generated by M5-brane instantons wrapped on M_6. This would break the shift symmetry to a discrete shift symmetry. In either case, the symmetry protects the axion from runaway behaviour.

Second, each Killing field on M_6 gives rise to a real massless scalar, which together transform in the adjoint on G. If G has rank 3 or greater (i.e. if M_6 has at least $U(1)^3$ isometry group - in other words, M_6 is toric) then the presence of these scalars can be understood from the fact that solution generating transformations can be used to generate continuous deformations of our background [21]. The moduli associated with these deformations correspond to massless scalar fields with exactly flat potentials and these must be at least a subset of the massless scalars arising from Killing fields on M_6. If G is simple then it acts transitively on the latter (since they transform in the adjoint of G), and hence they must all be moduli. This is the case for CP^3.

If G has an abelian factor then the massless scalar associated with the abelian generator is uncharged hence a runaway is possible. For the spaces listed in table 1, this happens

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Field	Type	m^2/c^2 $(k = 0, 1, 2, \ldots)$
Spin-2	real	$k(k+3)$
2-form	complex	$(k+2)^2$, $(k+4)^2$
	real	(k^2+5k+8)
1-form	complex	$(k+3)(k+4)$
	real	$(k+2)(k+3)$
	complex	$(k+2)^2$, $(k+4)^2$
	real	(k^2+5k+8)
	real	$k(k+1)$, $(k+3)(k+4)$
Scalar	complex	$(k+2)(k+4)$
	complex	$(k+3)(k+4)$
	real	$(k-1)(k+2)$, $(k+3)(k+6)$
	complex	$(k+1)(k+4)$ $\qquad k \geq ?$
	complex	$(k+2)^2$, $(k+4)^2$
	real	(k^2+5k+8)
	real	$k(k+1)$, $(k+3)(k+4)$
	real	0 (axion)

Table 4. The bosonic Kaluza-Klein spectrum for CP^3. The values of k have been shifted to take account of the restrictions in table 3: k is everywhere a non-negative integer except in the row corresponding to $\lambda^{(0,1)}_{(1,0)}$ (see main text).

only for $S^2 \times dP_3$ but we have already seen that this is unstable even at the classical level.

Third, if M_6 admits infinitesimal complex structure deformations then these will give complex massless scalars. These are present e.g. for $S^2 \times dP_{k>4}$ [15]. Since these are uncharged (because $dP_{k>4}$ has no continuous isometries), this suggests that these spaces will indeed suffer from a runaway instability.

Fourth, massless (complex) scalars arise if M_6 admits primitive harmonic $(2,1)$-forms. These will be gauge singlets since harmonic forms are invariant under continuous isometries. Hence such scalars could lead to a runaway instability. However, primitive harmonic $(2,1)$-forms do not arise for the spaces listed in table 1.

Finally, massless scalars arise if there are (transverse, primitive) $(1,1)$-form harmonics with eigenvalue $\lambda_{(1,1)} = 12c^2$. One would expect these to be charged in general so they will not generate a runaway. Such scalars are present for CP^3 and transform in the $[1,2,1]$ representation of SU(4).

3 The Kaluza-Klein spectrum

3.1 Decomposition of fields on M_6

On M_6, we can decompose a (p,q)-form into its primitive part and a non-primitive part:

$$X_{(p,q)} = X_{0(p,q)} + J \wedge X_{(p-1,q-1)}, \tag{3.1}$$

where a subscript 0 denotes a primitive form. We can further decompose a primitive form into a transverse part and exact pieces. Let $\Lambda_0^{(p,q)}$ denote the space of primitive (p,q)-forms. Define a map $\mathcal{F} : \Lambda_0^{(p-1,q)} + \Lambda_0^{(p,q-1)} \to \Lambda_0^{(p,q)}$ by

$$\mathcal{F}(Y_{0(p-1,q)} + Z_{0(p,q-1)}) = \left[\partial Y_{0(p-1,q)} + \bar{\partial} Z_{0(p,q-1)}\right]_0 , \tag{3.2}$$

where $[\ldots]_0$ denotes the primitive part. For given $X_{0(p,q)}$, choose $Y_{0(p-1,q)}$ and $Z_{0(p,q-1)}$ to minimize the inner product of $\mathcal{F}(Y_{0(p-1,q)} + Z_{0(p,q-1)})$ with $X_{0(p,q)}$. This results in the orthogonal decomposition

$$X_{0(p,q)} = \hat{X}_{(p,q)} + \left[\partial Y_{0(p-1,q)} + \bar{\partial} Z_{0(p,q-1)}\right]_0 , \tag{3.3}$$

where the hat denotes a form that is both primitive and transverse:

$$d^\dagger \hat{X}_{(p,q)} = 0 \Leftrightarrow \partial^\dagger \hat{X}_{(p,q)} = \bar{\partial}^\dagger \hat{X}_{(p,q)} = 0. \tag{3.4}$$

For example, we can decompose a general 1-form as

$$X_1 = \hat{X}_{(1,0)} + \hat{X}_{(0,1)} + \partial X + \bar{\partial} Y, \tag{3.5}$$

where X and Y are scalars. Using the above decomposition in two steps shows that a general 2-form can be decomposed into terms involving only primitive transverse forms as

$$X_2 = \hat{X}_{(2,0)} + \hat{X}_{(1,1)} + \hat{X}_{(0,2)} + \partial \hat{X}_{(1,0)} + \bar{\partial} \hat{X}_{(0,1)} + \partial \hat{Y}_{(0,1)} + \bar{\partial} \hat{Y}_{(1,0)} + \left[\partial \bar{\partial} Y\right]_0 + JX. \tag{3.6}$$

To avoid a proliferation of terms, we shall find it more convenient to work with n-forms, rather than (p,q)-forms, for most of our calculations. Note that, in the decomposition of a n-form X_n into (p,q)-forms of definite type, the individual (p,q)-forms will be transverse if, and only if, X_n is "doubly transverse", i.e.,

$$d^\dagger X = d^{c\dagger} X = 0, \tag{3.7}$$

where

$$d^c = -i(\partial - \bar{\partial}). \tag{3.8}$$

Hence we can rewrite the 1-form decomposition as (redefining X and Y)

$$X_1 = \hat{X}_1 + dX + d^c Y, \tag{3.9}$$

and the 2-form decomposition can be rewritten as

$$X_2 = \hat{X}_2 + d\hat{X}_1 + d^c \hat{Y}_1 + dd^c Y + JX, \tag{3.10}$$

where a hat on a n-form denotes that the form is primitive and doubly transverse. In the penultimate term of the 2-form decomposition, we have removed the square brackets from $dd^c Y$, which amounts to shifting the scalar X in the final term. Without the square brackets, the final two terms are no longer orthogonal but they are still linearly independent.

A 3-form X_3 can be decomposed as

$$X_3 = \hat{X}_3 + d\hat{X}_2 + d^c\hat{Y}_2 + dd^c\hat{X}_1 + J \wedge \left(\hat{Y}_1 + dX + d^cY\right). \tag{3.11}$$

Now consider a symmetric tensor h_{mn}. This can be decomposed into its hermitian and anti-hermitian parts:

$$h_{mn} = H_{mn} + A_{mn}, \qquad J_m{}^p J_n{}^q H_{pq} = H_{mn}, \qquad J_m{}^p J_n{}^q A_{pq} = -A_{mn}. \tag{3.12}$$

The hermitian part is equivalent to a $(1,1)$-form X:

$$H_{mn} = -X_{(1,1)mp} J^p{}_n. \tag{3.13}$$

X can be decomposed as described above. The anti-hermitian part A_{mn} can be split into its $(2,0)$ and $(0,2)$ parts. Consider the map $\mathcal{F}$ from $(1,0)$-forms to symmetric $(2,0)$ tensors defined by

$$\mathcal{F}(X_{(1,0)})_{mn} = \nabla^+_{(m} X_{n)}, \tag{3.14}$$

where $\nabla^\pm_m$ denote the projection of of ∇_m onto its $(1,0)$ and $(0,1)$ parts respectively. The space of $(2,0)$ symmetric tensors has the orthogonal decomposition $\mathrm{Im}(\mathcal{F}) + \mathrm{Ker}(\mathcal{F}^\dagger)$ and there is a similar decomposition for $(0,2)$ symmetric tensors so we can write

$$A_{mn} = \hat{A}_{mn} + \nabla^+_{(m} Y_{n)(1,0)} + \nabla^-_{(m} Y_{n)(0,1)}, \tag{3.15}$$

where $\hat{A}_{mn}$ is a transverse anti-hermitian tensor:

$$\nabla^m \hat{A}_{mn} = 0. \tag{3.16}$$

3.2 Decomposition of perturbation

Consider a small perturbation of the solution:

$$\delta g_{MN} = h_{MN}, \qquad \delta F_{MNPQ} = f_{MNPQ}. \tag{3.17}$$

The Bianchi identity implies $df = 0$ hence $f = da$ for some 3-form a.[6]

The AdS_5 components of the metric perturbation transform as a scalar on M_6 and can be expanded in eigenfunctions of the Hodge-de Rham Laplacian on M_6:

$$h_{\mu\nu}(x,y) = \sum_I h^I_{\mu\nu}(x) Y^I(y), \tag{3.18}$$

where $\Delta_6 Y^I = \lambda^I Y^I$. Decomposing $h^I_{\mu\nu}$ into transverse parts gives

$$h_{\mu\nu}(x,y) = \left(H_{\mu\nu}(x) + 2\nabla_{(\mu} H_{\nu)}(x) + 2\nabla_\mu \nabla_\nu H(x) + \frac{1}{5} T(x) g_{\mu\nu}\right) Y(y), \tag{3.19}$$

[6] M_6 admits at least one harmonic 4-form (i.e. $J \wedge J$) but we assume that f vanishes at infinity in AdS_5 so we don't need to include a contribution to f proportional to such a form.

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where $H_{\mu\nu}$ and H_μ are transverse. The I index and the summation are suppressed here, and henceforth. The gauge freedom $h_{MN} \to h_{MN} + 2\nabla_{(M}\xi_{N)}$ with $\xi_\mu(x,y) = -(H_\mu + \nabla_\mu H)Y$ and $\xi_m = 0$ can be used to fix the gauge

$$H_\mu = H = 0. \tag{3.20}$$

The mixed components of the metric perturbation can be decomposed as

$$h_{\mu m} = (Z_1 + dZ)_\mu \hat{Y}_{1m} + (Z_1^+ + dZ^+)_\mu (dY)_m + (Z_1^- + dZ^-)_\mu (d^cY)_m, \tag{3.21}$$

where Z_1 and $Z_1^\pm$ are transverse 1-forms in AdS_5 and $\hat{Y}_1$ is a doubly transverse 1-form on M_6.

As described above, the internal components of the metric perturbation can be decomposed into hermitian and anti-hermitian parts, and the hermitian part written in terms of a $(1,1)$-form:

$$h_{mn} = -X_{(1,1)mp}J^p{}_n + A_{mn}. \tag{3.22}$$

We decompose $X_{(1,1)}$ as described above:

$$\begin{aligned} X_{(1,1)} = {} & h(x)\hat{Y}_{(1,1)}(y) + 2N^{(1,0)}(x)\bar{\partial}\hat{Y}_{(1,0)}(y) + 2N^{(0,1)}(x)\partial\hat{Y}_{(0,1)}(y) \\ & + Q(x)dd^cY + \frac{1}{6}J_{mn}S(x)Y(y), \end{aligned} \tag{3.23}$$

where $\hat{Y}_{(1,1)}$ is primitive and transverse and $\hat{Y}_{(1,0)}$, $\hat{Y}_{(0,1)}$ are transverse. Note that $N^{(0,1)}$ and $N^{(1,0)}$ are (complex conjugate) scalar fields in AdS. It is convenient to suppress the indices on N and write this as

$$X_{(1,1)} = h\hat{Y}_{(1,1)} + Nd\hat{Y}_1 + Md^c\hat{Y}_1 + Qdd^cY + \frac{1}{6}J_{mn}SY, \tag{3.24}$$

where $Nd\hat{Y}_1 \equiv N^{(1,0)}d\hat{Y}_{(1,0)} + N^{(0,1)}d\hat{Y}_{(0,1)}$, $Md^c\hat{Y}_1 \equiv M^{(1,0)}d^c\hat{Y}_{(1,0)} + M^{(0,1)}d^c\hat{Y}_{(0,1)}$ and

$$M^{(1,0)} = -iN^{(1,0)}, \qquad M^{(0,1)} = iN^{(0,1)}. \tag{3.25}$$

We will sometimes write this as $M = \mp iN$ where the upper and lower signs refers to $(1,0)$ or $(0,1)$ respectively.

The anti-hermitian part A_{mn} can be decomposed as in (3.15):

$$A_{mn}(x,y) = A(x)\hat{Y}_{Tmn}(y) + B^{(1,0)}(x)\nabla^+_{(m}Y_{n)(1,0)}(y) + B^{(0,1)}(x)\nabla^-_{(m}Y_{n)(0,1)}(y), \tag{3.26}$$

where $\hat{Y}_{Tmn}$ denotes a transverse anti-hermitian tensor eigenfunction of the Lichnerowicz operator on M_6:

$$\Delta_L Y_{Tmn} \equiv -\nabla^2 Y_{Tmn} - 2R_{mpnq}Y_T^{pq} + 4c^2 Y_{Tmn} = \lambda_T Y_T. \tag{3.27}$$

A gauge transformation with $\xi_\mu = 0$ and $\xi_m = -(1/2)(B^{(1,0)}Y_{m(1,0)} + B^{(0,1)}Y_{m(0,1)})$ can be used to set

$$B^{(1,0)} = B^{(0,1)} = 0. \tag{3.28}$$

Note that this gauge transformation preserves (3.20). There is some residual gauge freedom:

$$\xi_\mu = k_\mu(x)Y(y), \qquad \xi_m = \alpha(x)V_m(y), \tag{3.29}$$

where k_μ and V_m are Killing vector fields in AdS_5 and M_6 respectively. As discussed above, the latter can always be written in terms of scalar harmonics [17, 18]

$$V_m = (d^cY)_m, \qquad \Delta_6 Y = 4c^2 Y. \tag{3.30}$$

The decomposition of the 3-form is:

$$\begin{aligned} a = {} & jY_{(3)} + k^+dY_{(2)} + k^-d^cY_{(2)} + (p_1 + dp)\wedge Y_{(2)} + \ell dd^cY_{(1)} + mJ\wedge Y_{(1)} \\ & +(q_1^+ + dq^+)\wedge dY_{(1)} + (q_1^- + dq^-)\wedge d^cY_{(1)} + (t_2 + dt_1)\wedge Y_{(1)} \\ & +(r_1 + dr)\wedge dd^cY + (u_2^+ + du_1^+)\wedge dY + (u_2^- + du_1^-)\wedge d^cY \\ & +n^+J\wedge dY + n^-J\wedge d^cY + (s_1 + ds)\wedge JY + (w_3 + dw_2)Y. \end{aligned} \tag{3.31}$$

We remind the reader that a sum over harmonics is understood, i.e., $jY_{(3)}$ stands for $j^I(x)Y^I_{(3)}(y)$. j, $k^\pm$ etc are scalars in AdS_5, p_1, $q_1^\pm$ etc are transverse vectors in AdS_5, $u_2^\pm$ etc are transverse 2-forms in AdS_5. We are also using the shorthand notation introduced above, e.g., $t_2\wedge Y_1$ stands for $t_2^{(1,0)}\wedge Y_{(1,0)} + t_2^{(0,1)}\wedge Y_{(0,1)}$ where $t_2^{(1,0)}$ and $t_2^{(0,1)}$ are complex conjugate 2-forms. In the final term, it will sometimes be convenient to rewrite the transverse 3-form w_3 in terms of a transverse 1-form v_1:

$$w_3 = \star_5 dv_1. \tag{3.32}$$

The 3-form a has gauge freedom $a \to a + d\Lambda$ for some 2-form Λ. However, the quantities in the above decomposition must arrange themselves into gauge-invariant combinations when we calculate the 4-form f. Computing f reveals that there is no loss of generality in imposing the gauge conditions

$$u_1^+ = q^+ = r = p = \ell = s = t_1 = r_1 = w_2 = 0. \tag{3.33}$$

We then have

$$\begin{aligned} f = {} & jdY_3 + k^-dd^cY_2 + mJ\wedge dY_1 + n^-J\wedge dd^cY \\ & +dj\wedge Y_3 - (p_1 - dk^+)\wedge dY_2 + dk^-\wedge d^cY_2 - (q_1^- + dq^-)\wedge dd^cY_1 \\ & +dm\wedge J\wedge Y_1 - (s_1 - dn^+)\wedge J\wedge dY + dn^-\wedge J\wedge d^cY \\ & +dp_1\wedge Y_2 + (t_2 + dq_1^+)\wedge dY_1 + dq_1^-\wedge d^cY_1 + (u_2^- + du_1^-)\wedge dd^cY + ds_1\wedge JY \\ & +dt_2\wedge Y_1 + (-w_3 + du_2^+)\wedge dY + du_2^-\wedge d^cY \\ & +dw_3Y \end{aligned} \tag{3.34}$$

There is some ambiguity in the decomposition of the AdS_5 fields into a transverse part and an exact part. An expression of the form $V_p + dV_{p-1}$, where V_p and V_{p-1} are transverse forms in AdS_5, is invariant under

$$V_{p-1} \to V_{p-1} + \delta V_{p-1}, \qquad V_p \to V_p - d\delta V_{p-1} \tag{3.35}$$

where δV_{p-1} is transverse and satisfies the equation of motion of a massless field in AdS_5:

$$\Delta\delta V_{p-1} = 0. \tag{3.36}$$

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3.3 The Maxwell equation

Perturbing the Maxwell equation gives

$$\star\, d \star f + \star d\delta(\star)\bar{F} = \star\left(f \wedge \bar{F}\right), \tag{3.37}$$

where a bar refers to the unperturbed solution and $\delta(\star)\bar{F}$ denotes the change in $\star\bar{F}$ resulting from the metric perturbation. In evaluating this equation, the following results are useful. Let X_p and Y_q denote a p-form in AdS_5 and a q-form in M_6 respectively. Then

$$\star\,(X_p \wedge Y_q) = (-)^{pq}(\star_5 X_p) \wedge (\star_6 Y_q), \tag{3.38}$$

Now take $q = 4 - p$ with $X_p \wedge Y_{4-p}$ a typical term in the decomposition of the Maxwell perturbation f. On the l.h.s. of the Maxwell equation we will encounter terms of the form

$$\star\, d \star (X_p \wedge Y_{4-p}) = -\left(d_5^\dagger X_p\right) \wedge Y_{4-p} + (-)^{p+1} X_p \wedge d_6^\dagger Y_{4-p}. \tag{3.39}$$

The metric perturbation also enters the l.h.s. of the Maxwell equation. We find

$$\begin{aligned}\star\, d\delta(\star)F \;=&\; -cJ \wedge d_6^c h^M_M + 4cd_6^c X_{(1,1)} + cJ \wedge d_6^\dagger X_{(1,1)} \\ &-2cd_5^\dagger X_1' J \wedge J \cdot Y_1' + 2cX_1' \wedge d_6^c(J \cdot Y_1') + 2cX_1' \wedge J \wedge d_6^\dagger(J \cdot Y_1'),\end{aligned} \tag{3.40}$$

where X_1', Y_1' denote the various terms arising from the mixed components $h_{\mu m}$, i.e., $h_{\mu m}$ is a sum of terms of the form $(X_1')_\mu (Y_1')_m$, and the corresponding sum should be understood in the above expression.

Using these results, the Maxwell equation decomposes as follows. The $\mu\nu\rho$ components give

$$\lambda \star du_2^+ + 2c\lambda u_2^- + d\left[(\Delta + \lambda)v_1 + 2c\lambda u_1^- + 6cs_1\right] = 0. \tag{3.41}$$

The $\mu\nu m$ components describe 1-forms on M_6. These can be decomposed into a transverse 1-form part, arising from terms proportional to $\hat{Y}_{1m}$ and scalar parts proportional to dY and d^cY respectively. The transverse $(1,0)$-form part is (t_2 denotes $t_2^{(1,0)}$ etc)

$$(\Delta + \lambda_1)t_2 + \lambda_1 dq_1^+ + 2ic \star dt_2 = 0. \tag{3.42}$$

The transverse $(0,1)$-form part is the complex conjugate of this. Now $\lambda_1 \neq 0$ (see above) so acting on this equation with $d^\dagger$ gives $\Delta q_1^+ = 0$. This implies that q_1^+ can be gauged away using the freedom (3.35), i.e., we can absorb q_1^+ into t_2. So we set $q_1^+ = 0$ henceforth. This leaves

$$(\Delta + \lambda_1)t_2 + 2ic \star dt_2 = 0. \tag{3.43}$$

The terms proportional to dY give the same equation as d acting on (3.41), while the terms proportional to d^cY give

$$(\Delta + \lambda)u_2^- - 2c \star du_2^+ + d\left[\lambda u_1^- + s_1 - 2cv_1\right] = 0, \qquad \lambda \neq 0. \tag{3.44}$$

The restriction $\lambda \neq 0$ arises because otherwise $d^cY = 0$. The 1-form and 2-form parts of this equation and equation (3.41) can be decoupled using the gauge freedom (3.35). Consider a transformation $u_1^- \to u_1^- + \delta u_1^-$, $v_1 \to v_1 + \delta v_1$, $u_2^- \to u_2^- - d\delta u_1^-$, $u_2^+ \to u_2^+ + \delta u_2^+$, with $\Delta\delta v_1 = \Delta\delta u_1^- = 0$ and δu_2^+ is defined by $d\delta u_2^+ = \star d\delta v_1$. This leaves the 4-form invariant. Acting with $d^\dagger$ on the above equations implies that the square brackets in both are annihilated by Δ. This implies that we can choose δu_1^- and δv_1 to make these brackets vanish. Hence the 2-form and 1-form parts of these equations decouple. The 2-form equations give

$$\star\, du_2^+ + 2cu_2^- = 0, \qquad \lambda \neq 0, \tag{3.45}$$

and (after using this equation to eliminate u_2^+),

$$(\Delta + \lambda + 4c^2)u_2^- = 0, \qquad \lambda \neq 0. \tag{3.46}$$

Hence u_2^- is a massive 2-form field with $m^2 = \lambda + 4c^2$, and u_2^+ is not independent. If $\lambda = 0$ then $u_2^\pm$ drop out of the expression for f and are therefore unphysical. The 1-form equations are

$$(\Delta + \lambda)v_1 + 2c\lambda u_1^- + 6cs_1 = 0, \tag{3.47}$$

$$\lambda u_1^- + s_1 - 2cv_1 = 0, \qquad \lambda \neq 0. \tag{3.48}$$

The μmn components of the Maxwell equation correspond to 2-forms on M_6, which can be decomposed into irreducible pieces as described above. The terms proportional to $\hat{Y}_2$ give $\Delta p_1 + \lambda_2(p_1 - dk^+) = 0$, which implies that $\Delta k^+ = 0$ so we can absorb k^+ into p_1 using the residual freedom (3.35). This leaves

$$(\Delta + \lambda_2)\, p_1 = 0. \tag{3.49}$$

Hence p_1 is a vector field with $m^2 = \lambda_2$. The terms proportional to $d\hat{Y}_1$ vanish when we use $q_1^+ = 0$. The terms proportional to $d^c\hat{Y}_1$ give, for a $(1,0)$-form $\hat{Y}_1$ (so q_1^- denotes $q_1^{-(1,0)}$ etc, $(0,1)$-forms give the complex conjugate of this) a 1-form part[7]

$$(\Delta + \lambda_1)\, q_1^- + 2icZ_1 = 0, \tag{3.50}$$

and a scalar part

$$\lambda_1 q^- - m + 2icZ = 0. \tag{3.51}$$

The terms proportional to dd^cY give (NB $dd^cY = 0$ if, and only if, $\lambda = 0$) a 1-form part

$$\Delta u_1^- - s_1 - 2cZ_1^- = 0 \qquad \lambda \neq 0, \tag{3.52}$$

and a scalar part

$$n^+ = 2cZ^- \qquad \lambda \neq 0. \tag{3.53}$$

The terms proportional to JY give 1-form part

$$\Delta s_1 + \lambda s_1 + 2c\lambda Z_1^- - 2c\Delta v_1 = 0. \tag{3.54}$$

[7]The split into 1-form and scalar parts uses the freedom (3.35) as described above. Strictly speaking, this can only be done once we have the complete set of equations governing these fields, but we shall anticipate the final result and split the equations as we encounter them.

and the scalar part reproduces (3.53).

Finally, we consider the mnp components of the Maxwell equation. These transform as a 3-form on M_6, which can be decomposed as described above. Doing so, the terms proportional to $\hat{Y}_3$ give

$$(\Delta + \lambda_3)j = 0, \tag{3.55}$$

so the scalar field j has $m^2 = \lambda_3$. The terms proportional to $d\hat{Y}_2$ vanish (using $k^+ = 0$). Terms proportional to $d^c\hat{Y}_2$ give (if $\lambda_2 = 0$ then $\hat{Y}_2$ is harmonic so $d^c\hat{Y}_2 = 0$)

$$(\Delta + \lambda_2)k^- - 4ch = 0 \qquad \lambda_2 \neq 0. \tag{3.56}$$

Terms proportional to $dd^c\hat{Y}_1$ give

$$\Delta q^- + m - 4cN = 0. \tag{3.57}$$

Terms proportional to $J \wedge \hat{Y}_1$ give (this comes from the $(1,0)$ part of $\hat{Y}_1$, the $(0,1)$ part gives the complex conjugate)

$$(\Delta + \lambda_1)m - 4c\lambda_1 N - 2ic\Delta Z = 0, \tag{3.58}$$

Terms proportional to $J \wedge dY$ vanish upon using (3.53). Terms proportional to $J \wedge d^cY$ give

$$(\Delta + \lambda)n^- + cT - \frac{1}{3}cS - 2c\lambda Q + 2c\Delta Z^+ = 0 \qquad \lambda \neq 0. \tag{3.59}$$

3.4 The Einstein equation

The perturbed Einstein equation is

$$\delta R_{MN} = \delta S_{MN}, \tag{3.60}$$

where

$$\delta R_{MN} = -\frac{1}{2}\left(\nabla_5^2 + \nabla_6^2\right) h_{MN} - \frac{1}{2}\nabla_M \nabla_N h^P_P + \nabla_{(M}\nabla^P h_{N)P} - \bar{R}_{MPNQ}h^{PQ} + \bar{R}^P_{(M}h_{N)P}, \tag{3.61}$$

and

$$\delta S_{MN} = \frac{1}{12}\Big[2f_{(M|PQR|}\bar{F}_{N)}{}^{PQR} - 3\bar{F}_{MPRS}\bar{F}_{NQ}{}^{RS}h^{PQ} - \frac{1}{12}h_{MN}\bar{F}_{PQRS}\bar{F}^{PQRS} - \frac{1}{12}\bar{g}_{MN}\left(2f_{PQRS}\bar{F}^{PQRS} - 4\bar{F}_{PRST}\bar{F}_Q{}^{RST}h^{PQ}\right)\Big]. \tag{3.62}$$

Evaluating the $\mu\nu$ components and decomposing into irreducible parts gives transverse traceless tensor part

$$-\nabla_5^2 H_{\mu\nu} + (\lambda - c^2)H_{\mu\nu} = 0, \tag{3.63}$$

so $H_{\mu\nu}$ is a massive spin-2 field, for $\lambda = 0$ we obtain the massless AdS_5 graviton. The 1-form part is

$$\nabla_{(\mu}Z^+_{1\nu)} = 0 \qquad \lambda \neq 0, \tag{3.64}$$

which implies that Z_1^+ can be gauged away using the residual gauge invariance (3.29). (If $\lambda = 0$ then Z_1^+ drops out of $h_{\mu m}$ so is unphysical.) Hence we set $Z_1^+ = 0$ henceforth. Terms of the form $\nabla_\mu \nabla_\nu$(scalar) give

$$\lambda Q + \frac{1}{2} S + \frac{3}{10} T + \lambda Z^+ = 0, \tag{3.65}$$

and terms proportional to $\bar{g}_{\mu\nu}$ give

$$\frac{1}{10}\left(\Delta + \lambda + 4c^2\right) T + \frac{4}{3}\left(c\lambda n^- - c^2 S - 2c^2 \lambda Q\right) = 0. \tag{3.66}$$

The μm components of the Einstein equation can be decomposed into transverse 1-form and scalar parts on M_6. These can then be decomposed into transverse 1-form and scalar parts on AdS_5. The transverse $(1,0)$-form part gives AdS_5 1-form equation

$$\frac{1}{2}(\Delta + \lambda_1 + 4c^2) Z_1 - ic\lambda_1 q_1^- = 0, \tag{3.67}$$

and the AdS_5 scalar part is

$$\frac{1}{2}\lambda_1 (Z + iN - 2icq^-) + 2icm + 2c^2 Z = 0. \tag{3.68}$$

From terms proportional to dY we obtain vanishing AdS_5 1-form part (using $Z_1^+ = 0$). The scalar part is

$$\lambda Q + \frac{5}{6} S + \frac{4}{5} T - 4cn^- - 4c^2 Z^+ = 0 \qquad \lambda \neq 0. \tag{3.69}$$

From terms proportional to $d^c Y$ we obtain 1-form part

$$\left(\Delta + \lambda + 4c^2\right) Z_1^- + 4cs_1 = 0 \qquad \lambda \neq 0, \tag{3.70}$$

and scalar part

$$(\lambda + 4c^2) Z^- = 4cn^+ \qquad \lambda \neq 0. \tag{3.71}$$

Combining this with (3.53) gives

$$n^+ = Z^- = 0, \tag{3.72}$$

unless $\lambda = 0$ or $\lambda = 4c^2$. In the former case, n^+ and Z^- are unphysical. The latter case corresponds to a harmonic Y for which $d^c Y$ is a Killing field on M_6. In this case, we can use the residual gauge freedom (3.29) to set $Z^- = 0$ so equation (3.53) gives $n^+ = 0$. Hence equation (3.72) is satisfied in general.

Next consider the mn components of the Einstein equation, which only involve AdS_5 scalars. First we decompose these into hermitian and anti-hermitian parts. The transverse anti-hermitian part gives

$$\left(\Delta + \lambda_T - 4c^2\right) A = 0, \tag{3.73}$$

where λ_T is an eigenvalue of the Lichnerowicz operator on M_6 corresponding to tranverse anti-hermitian modes. The anti-hermitian part also has transverse 1-form, and scalar parts. The transverse $(1,0)$-form part is

$$\Delta Z - i\lambda_1 N = 0. \tag{3.74}$$

After using $Z^- = 0$, the scalar part, proportional to $\nabla^\pm_m \nabla^\pm_n Y$ gives

$$\Delta Z^+ + \frac{1}{3}S + \frac{1}{2}T = 0 \qquad \lambda \neq 0. \tag{3.75}$$

The hermitian part of the mn Einstein equation can be converted to a $(1,1)$-form and decomposed as described above. The transverse primitive part gives

$$\left(\Delta + \lambda_{(1,1)} + 4c^2\right) h^{(1,1)} - 4c\lambda_{(1,1)} k^{-(1,1)} = 0. \tag{3.76}$$

The transverse vector part gives

$$\left(\Delta + \lambda_1 + 4c^2\right) N - 2cm = 0. \tag{3.77}$$

The scalar part proportional to dd^cY gives

$$\left(\Delta + \lambda + 4c^2\right) Q - 4cn^- = 0 \qquad \lambda \neq 0. \tag{3.78}$$

The scalar part proportional to JY gives

$$\left(\Delta + \lambda + 12c^2\right) S - 8c\lambda n^- + 16c^2\lambda Q = 0. \tag{3.79}$$

3.5 The mass spectrum

In this section we shall diagonalize the above equations to determine the full Kaluza-Klein spectrum.

3.5.1 Symmetric tensor/scalar modes

This sector contains just the real, transverse, traceless, symmetric tensor field $H_{\mu\nu}$ with equation of motion (3.63). For $\lambda = 0$ this gives the AdS_5 graviton. $\lambda > 0$ gives massive spin-2 fields.

3.5.2 2-form/1-form modes

In this sector we have the complex field t_2. The equation of motion is (3.43). To obtain the mass associated with this field, we note that a complex 2-form in AdS_5 has a *first* order equation of motion [22], so t_2 is actually equivalent to two complex 2-form fields. Equation (3.43) can be decomposed into first order equations by defining

$$Z_2 = t_2 + ia \star_5 dt_2, \tag{3.80}$$

and seek a so that $\star_5 dZ_2 \propto Z_2$. This requires $\lambda_1 a^2 + 2ac - 1 = 0$, so there are two solutions: $\lambda_1 a_\pm = \mp\sqrt{\lambda_1 + c^2} - c$. Hence there are two linearly independent solutions $Z_2^\pm$. Obviously t_2 can be written as a linear combination of these two fields. We then have

$$\star\, dZ^\pm_{(2)} = -ia_\pm \lambda_1 Z_2^\pm. \tag{3.81}$$

This is the equation of motion of a complex 2-form with mass given by $m_\pm^2 = (a_\pm\lambda_1)^2$ (see e.g. [22]). To see this, note that acting with $\star d$ gives

$$(\Delta_5 + (a_\pm\lambda_1)^2) Z_2^\pm = 0. \tag{3.82}$$

Hence we have two complex 2-form fields of definite mass, namely $Z^\pm_{(2)}$, with masses given by

$$m_\pm = \sqrt{\lambda_1 + c^2} \pm c. \tag{3.83}$$

As discussed above, $\lambda_1 > 0$ so these fields are both massive.

3.5.3 2-form/scalar modes

In this section we have the real fields $u_2^{\pm}$. u_2^+ is given in terms of u_2^- by equation (3.45), and u_2^- has equation of motion (3.46). Hence this sector contains a single real 2-form with $m^2 = \lambda + 4c^2$, $\lambda > 0$.

3.5.4 1-form/2-form modes

The only field in this sector is p_1, with equation of motion (3.49). This can be decomposed into the complex field $p_1^{(2,0)}$ (with complex conjugate $p_1^{(0,2)}$) with $m^2 = \lambda_{(2,0)}$ and a real field $p_1^{(1,1)}$ (since we can take $(1,1)$-form eigenfunctions of Δ_6 to be real) with $m^2 = \lambda_{(1,1)}$. Note that (primitive, transverse) harmonic 2-forms give rise to massless 1-forms in AdS_5.

3.5.5 1-form/1-form modes

In this sector we have the complex 1-form fields q_1^- and Z_1 (or, more precisely, $q_1^{-(1,0)}$ and $Z_1^{(1,0)}$) with equations of motion (3.50), (3.67). (We saw above that q_1^+ can be gauged away.) Diagonalizing gives the masses as

$$m^2 = \lambda_1 + 2c^2 \pm \sqrt{(\lambda_1 + 2c^2)^2 - \lambda_1^2}. \tag{3.84}$$

These fields are all massive (because $\lambda_1 > 0$).

3.5.6 1-form/scalar modes

The fields in this sector are v_1, u_1^-, s_1 and Z_1^-. (We saw above that equation (3.64) implies that Z_1^+ can be gauged away.) These fields are real. They are governed by the equations of motion (3.47), (3.52), (3.54), (3.70) and the constraint (3.48).

Consider first the case $\lambda = 0$. In this case, the only physical fields are s_1 and v_1 and the only non-trivial equations are (3.47), which gives $\Delta v_1 + 6cs_1 = 0$, and (3.54), which gives $\Delta(s_1 - 2cv_1) = 0$. Combining these gives

$$\Delta(s_1 - 2cv_1) = 0, \qquad \left(\Delta + 12c^2\right) s_1 = 0, \qquad \lambda = 0. \tag{3.85}$$

Hence $s_1 - 2cv_1$ is massless and s_1 has $m^2 = 12c^2$. Recall that v_1 arises from the AdS_5 components of the M-theory 3-form via $w_3 = \star dv_1$. Hence the massless field we have found here is essentially the Kaluza-Klein zero mode of the M-theory 3-form. This massless 3-form can be dualized to a scalar via $d(w_3 - (1/2c) \star ds_1) = \star d\sigma$. This scalar has a gauge invariance $\sigma \sim \sigma + \text{constant}$.

Now consider the case $\lambda \neq 0$. It can be verified that the constraint equation (3.48) is consistent with the four equations of motion. This constraint can be used to eliminate, say, s_1. This leaves three fields. The equations of motion can be combined to give

$$\left(\Delta + \lambda + 4c^2\right)\left(v_1 - Z_1^-\right) = 0 \qquad \lambda \neq 0, \tag{3.86}$$

so $v_1 - Z_1^-$ is a field with $m^2 = \lambda + 4c^2$. The remaining two mass eigenstates can be identified by setting $\mathcal{U}_1 = u_1^- + \alpha v_1 + \beta(Z_1^- - v_1)$ and choosing α, β so that $(\Delta + m^2)\mathcal{U}_1 = 0$

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for some m. This gives $\beta = 1/(2\lambda\alpha + 2c)$, $\alpha = (-3c \mp \sqrt{9c^2 + 4\lambda})/(2\lambda)$. Denote the corresponding linear combinations as $\mathcal{U}_{1\pm}$. Their masses are

$$m_\pm^2 = \lambda + 6c^2 \pm \sqrt{(\lambda + 6c^2)^2 - \lambda(\lambda - 4c^2)} \qquad \lambda \neq 0. \tag{3.87}$$

Hence, for $\lambda = 4c^2$, $\mathcal{U}_{1-}$ is a massless real vector field. But scalar modes with $\lambda = 4c^2$ are in one-to-one correspondence with Killing vector fields on M_6. Hence these massless vectors must be the Kaluza-Klein gauge bosons.

3.5.7 Scalar/anti-hermitian tensor modes

A symmetric anti-hermitian tensor can be decomposed into $(2,0)$ and $(0,2)$ parts, so we have two complex conjugate fields $A^{(2,0)}$ and $A^{(0,2)}$, with equation of motion given by (3.73). Hence we have $m^2 = \lambda_T - 4c^2$. This can be seen to be non-negative using the following standard argument that relates anti-hermitian eigenfunctions of the Lichnerowicz operator to complex structure deformations [17].

Consider an anti-hermitian $(2,0)$ tensor eigenfunction $\hat{Y}_{mn}$ with eigenvalue λ. Raising an index, we have a tensor $\hat{Y}_n^m$ which can be regarded as a $(0,1)$-form taking values in $T^{1,0}M_6$, the holomorphic tangent space of M_6. For a $(0,q)$-form ω taking values in $T^{1,0}M_6$ we define

$$(\bar{\partial}\omega)^m_{np_1\dots p_q} = (q+1)\nabla^-_{[n}\omega^m_{p_1\dots p_q]}, \tag{3.88}$$

where ∇^-_m denote the $(0,1)$ part of ∇_m. For any two such forms ω, ν we define the obvious inner product

$$(\omega, \nu) = \frac{1}{q!}\int \omega^m_{n_1\dots n_p}\, g_{mm'} g^{n_1n_1'} \dots g^{n_pn_p'}\, \bar{\nu}^{m'}_{n_1'\dots n_p'}. \tag{3.89}$$

We can then defines the adjoint $\bar{\partial}^\dagger$. Transversality implies that $(\bar{\partial}^\dagger Y)^m = 0$. Now define the Laplacian acting on $(0,q)$-forms taking values in $T^{1,0}M_6$ by $\Delta_{\bar{\partial}} \equiv 2\left(\bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}\right)$. Acting on Y, we find that

$$(\Delta_{\bar{\partial}}Y)^m_n = [(\Delta_L - 4c^2)Y]^m_n = (\lambda_T - 4c^2)Y^m_n. \tag{3.90}$$

Hence the mass of the complex scalar in this sector is given by

$$m^2 = \lambda^{(1,0)}_{(0,1)}, \tag{3.91}$$

where $\lambda^{(1,0)}_{(0,1)}$ denotes the eigenvalues of $\Delta_{\bar{\partial}}$. These are manifestly non-negative. Modes with $m = 0$ correspond to infinitesimal deformations of the complex structure of M_6.

3.5.8 Scalar/3-form modes

The only field here is j, or, more precisely, the complex scalar $j^{(2,1)}$. The equation of motion is (3.55) so $j^{(2,1)}$ has $m^2 = \lambda_{(2,1)}$. There are no transverse $(3,0)$-forms hence there is no $j^{(3,0)}$ part.

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3.5.9 Scalar/2-form modes

The fields in this sector are h and k^-. Their equations of motion are given by equations (3.56) and (3.76). Now h is associated with $(1,1)$-forms, i.e., $h^{(2,0)} = h^{(0,2)} = 0$. Hence (3.56) gives

$$(\Delta + \lambda_{(2,0)})k^{-(2,0)} = 0 \qquad \lambda_{(2,0)} \neq 0, \tag{3.92}$$

and $k^{-(0,2)}$ is the complex conjugate of $k^{-(2,0)}$. So $k^{-(2,0)}$ is a complex massive scalar field with $m^2 = \lambda_{(2,0)} > 0$.

For the $(1,1)$-forms, we have to diagonalize equations (3.56) and (3.76), which was discussed in section 2.2.2.

3.5.10 Scalar/1-form modes

In this sector we have the complex fields m, q^-, Z and N. (More precisely: $m^{(1,0)}$, $q^{-(1,0)}$ etc.) These satisfy the equations of motion (3.57), (3.58), (3.74), (3.77) and the constraints (3.51), (3.68). These constraints are compatible with the equations of motion and can be used to eliminate, say, q^- and Z, leaving two fields m, N. The equations of motion for m and N are (3.77) and

$$(\Delta + \lambda_1)m - 2c\lambda_1 N = 0. \tag{3.93}$$

Diagonalizing gives the masses as

$$m^2 = \lambda_1 + 2c^2 \pm \sqrt{(\lambda_1 + 2c^2) - \lambda_1^2}. \tag{3.94}$$

Since $\lambda_1 > 0$, these two fields are massive.

3.5.11 Scalar/scalar modes

This sector contains the real fields n^-, S, Z^+, Q, T (we saw above that $n^+ = Z^- = 0$). The equations of motion are (3.59), (3.66), (3.75), (3.78), (3.79) and there are two constraints (3.65), (3.69). It can be checked that the constraints are consistent with the equations of motion.

If $\lambda = 0$ then the only physical modes are S and T, obeying the equations of motion (3.66), (3.79) and the constraint (3.65). The constraint can be used to eliminate, T, leaving

$$(\Delta + 12c^2)S = 0 \qquad \lambda = 0, \tag{3.95}$$

so for $\lambda = 0$ we have a single field with $m^2 = 12c^2$.

Now assume $\lambda > 0$. The constraints can be used to eliminate S and T, leaving three fields. The other equations can be rearranged to give

$$(\Delta + \lambda + 4c^2)(Q + Z^+) = 0 \qquad \lambda \neq 0 \tag{3.96}$$

hence $Q + Z^+$ is a field with $m^2 = \lambda + 4c^2$. The remaining two linear combinations with definite mass can be identified by setting $\mathcal{V} = n^- + \alpha Z^+ + \beta(Q + Z^+)$ and choosing α, β so that the equations of motion imply $(\Delta + m^2)\mathcal{V} = 0$. This requires $\beta = \lambda/(3\alpha + 3c)$ and

$\alpha = (1/2)(-c \pm \sqrt{4\lambda + 9c^2})$, corresponding to two linear combinations $\mathcal{V}_\pm$. The masses are given by

$$m_\pm^2 = \lambda + 6c^2 \pm \sqrt{(\lambda + 6c^2)^2 - \lambda(\lambda - 4c^2)} \qquad \lambda \neq 0. \tag{3.97}$$

Scalar modes with $\lambda = 4c^2$ give a massless field $\mathcal{V}_-$. As discussed above, such modes are in one-to-one correspondence with Killing vector fields of M_6.

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4 Conventions

We use a positive signature metric. The bosonic action for eleven-dimensional supergravity is given by

$$16\pi G S = \int d^{11}x\sqrt{-g}R + \int \left(-\frac{1}{2}F \wedge \star F + \frac{1}{6}A \wedge F \wedge F\right), \tag{4.1}$$

where $F = dA$ is the 4-form. The equations of motion are

$$R_{MN} = \frac{1}{12}\left(F_{MPQR}F_N{}^{PQR} - \frac{1}{12}g_{MN}F_{PQRS}F^{PQRS}\right), \qquad d \star F = \frac{1}{2}F \wedge F. \tag{4.2}$$

The orientation is fixed by specifying the 11d volume form

$$\eta_{11} = \eta_5 \wedge \eta_6, \tag{4.3}$$

where η_5 and η_6 are the volume forms of AdS_5 and M_6 respectively. η_6 is related to the Kähler form by

$$\eta_6 = 6J \wedge J \wedge J. \tag{4.4}$$

On M_6 we have

$$d_6^\dagger = \star_6 d_6 \star_6, \tag{4.5}$$

and the Laplacian is

$$\Delta_6 = d_6 d_6^\dagger + d_6^\dagger d_6. \tag{4.6}$$

We also have the Dolbeault operators ∂, $\bar\partial$ such that $d_6 = \partial + \bar\partial$. We can define an exterior derivative d_6^c using $J_m{}^n\nabla_n$, or, equivalently,

$$d_6^c = -i(\partial - \bar\partial). \tag{4.7}$$

On AdS_5, for a p-form X, we define

$$d_5^\dagger X_p = (-)^{p+1} \star_5 d_5 \star_5 X_p \tag{4.8}$$

and the wave operator is

$$\Delta = d_5 d_5^\dagger + d_5^\dagger d_5. \tag{4.9}$$

A free p-form field of mass m has equation of motion

$$\left(\Delta + m^2\right) X_p = 0. \tag{4.10}$$

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